Knots and surfaces in space

We can imagine a *curve* in space as a piece of string (or rope), possibly twisted and intertwined (knotted), suspended in space. If we pull at its two ends, it may straighten out or develop tight knots. However, by carefully untangling it before pulling we can always continuously deform it to a straight piece of string representing a line segment. This means that up to allowing continuous deformations, any piece of string is unknotted, i.e., it represents a line segment. The situation changes if we bring the two ends of the string together and fuse them – this operation produces a closed curve which we call a *knot*. If the initial string represents a line segment (lying in a plane) and we bring the two ends together and fuse them all the while keeping the string in the plane, the resulting knot can always be continuously deformed to a round circle which we also call the *unknot*, as it has no knotting. But starting with a twisted and knotted string this operation may produce a more interesting knot that is (or at least seems to be) different from the unknot in the sense that one cannot continuously deform it to the unknot. The basic

FIGURE 1. Left: a twisted curve in space. Middle: moving the ends of the curve together. Right: the resulting knot is a trefoil.

question concerning knots therefore is whether a given knot is the unknot, and more generally when two knots are equivalent in the sense that one can be continuously deformed into the other. When posed for more general objects than knots, this is a central question of topology. But the example of knots also demonstrates another important concept: even though any knot is the image of a circle (string with the two ends fused together), the variation in the knots comes from the way they sit in space. So not just the object itself, of interest may also be its position in space.

A knot K has an important property: it looks the same close to any point $x \in K$. Indeed, the set of all points N_x in K sufficiently close to x can be continuously deformed to an interval (a line segment) with x as its center. Since the same is true of any point on the real line $\mathbb{R}$, we say that K locally looks like $\mathbb{R}$. Objects with the property that they look the same around any point are called *homogeneous*. Any closed curve is therefore an example of a homogeneous space, but not every curve is homogeneous: for an end point x of a line segment no set of nearby points U_x can be continuously deformed to an interval with x as its center. Similarly, an object in the shape of letter X does not look like a line close to its central point (in fact, since from this point emanate 4 line segments, X is not even a curve).

The main class of objects that will be of interest to us are surfaces which locally look like the plane $\mathbb{R}^2$ (if this holds at all points a surface is homogeneous and is called a closed surface) or a half plane (points that lie on the dividing line on the plane cutting it into two half planes are analogous to the end points of an interval and are called *boundary points*). Examples of closed surfaces are the surface of a ball – called a *sphere*, and the surface of a doughnut (or the surface of a tire inner tube) – called a *torus*. This shows that unlike for the curves, where any closed curve is the image of a circle, there is more than one closed surface. In the course we will try to understand all the possible surfaces, including those having boundary points. Note that

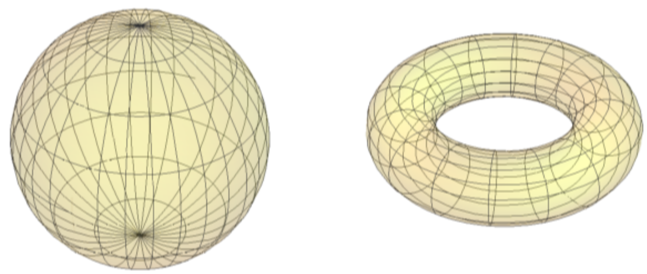

FIGURE 2. Left: a sphere. Right: a torus.

the boundary points always form (collections of) knots. In order to recognize different surfaces we will need to find properties of surfaces that are preserved under continuous deformations. An important such property is demonstrated by the distinction between the two bands in the picture below. They can both be constructed from a long thin strip of paper by gluing together the two short sides of the rectangle. The difference between the two is that in the second one we introduce a half twist in the band. How do we tell them apart? What happens if we put two

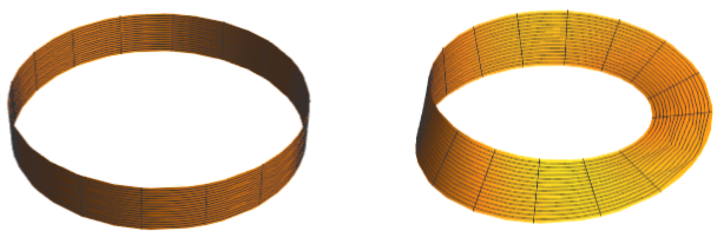

FIGURE 3. Left: a cylinder. Right: a Möbius band.

half twists in the band?

In the course we will explore some properties of knots and their connections to surfaces. We will try to understand surfaces and in the process of doing so will encounter graphs on surfaces: these are finite collections of distinguished points (called vertices) with arcs (called edges) connecting them that can only intersect in vertices. Through this combinatorial structure we will be able to assign a number to a surface, called its *Euler characteristic*, that is a very powerful tool for recognizing surfaces.

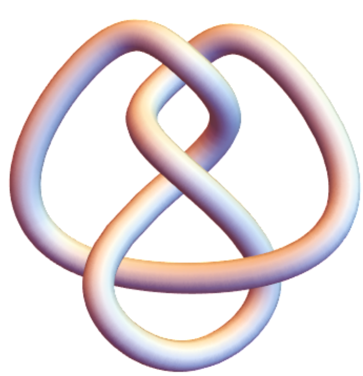

FIGURE 4. A knot? A torus? A knotted torus!