Number Theory

We will start with the notion of a formal power series. While it is important that identities like

$$\frac{1}{1-x} = \sum_{n\geqslant 0} x^n \qquad (|x| < 1)$$

and

$$e^x = \sum_{n\geqslant 0} \frac{x^n}{n!}$$

are equalities of real-valued functions, it is also important to be able to treat such equations as purely algebraic identities. Doing so allows one to prove, for example, that $e^{x+y} = e^x e^y$, this identity being a simple consequence of the binomial theorem.

Next we will introduce the Bernoulli numbers b_k. They are defined by the identity of formal power series

$$\frac{t}{e^t - 1} = \sum_{n\geqslant 0} b_k \frac{t^k}{k!}.$$

We will also consider the Bernoulli polynomials $B_k(x)$, which are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{n\geqslant 0} B_k(x) \frac{t^k}{k!}.$$

The Bernoulli numbers and Bernoulli polynomials have many remarkable properties. For example, the finite sum

$$1 + 2^k + 3^k + \cdots + N^k$$

has a simple expression in terms of Bernoulli polynomials, and the infinite sum

$$1 + 2^{-k} + 3^{-k} + 4^{-k} + \ldots$$

has a simple expression in terms of Bernoulli numbers when k is even. The latter sum is actually the value at $s = k$ of the Riemann zeta function $\zeta(s)$, a function which is at the heart of the most famous unsolved problem in mathematics.