4. Uniform Distribution Theory (19 July 2019).

1. **Definition 1.** An infinite sequence ξ_j, $j = 1, 2, 3, ...$ of points from the unit interval $[0, 1)$ is called uniformly distributed (U.D.) if the following holds. For every $\gamma \in [0, 1)$ and for every $q \in \mathbb{Z}_+$ the quantity

$$N_q(\gamma) = |\{n \in \mathbb{Z}_+ : \ \ n \le q, \{\xi_n\} \le \gamma\}|$$

satisfies

$$\lim_{q\to\infty} \frac{N_q(\gamma)}{q} = \gamma,$$

or

$$N_q(\gamma) = \gamma q + o(q), \ \ q \to \infty.$$

2. **Definition 2.** Consider a finite sequence $\Xi = \{\xi_1, ..., \xi_q\}$. *Discrepancy* of this sequence is defined as

$$D(\Xi) = \sup_{\gamma\in[0,1)} |N_q(\gamma) - \gamma q|.$$

3. What can you say about the discrepancy of the sequence
a. $\frac{0}{q}, \frac{1}{q}, \frac{2}{q}, ..., \frac{q-1}{q}$?
b. $\frac{a\cdot 0}{q}, \frac{a\cdot 1}{q}, \frac{a\cdot 2}{q}, ..., \frac{a\cdot(q-1)}{q}$, where $(a, q) = 1$?

4. Obvious proposition. For an infinite sequence ξ_j, $j = 1, 2, 3, ...$ consider its beginning $\Xi_q = \{\xi_1, ..., \xi_q\}$. Then if

$$\lim_{q\to\infty} \frac{D_q}{q} = 0, \ \ \ D_q = D(\Xi_q),$$

then the sequence is U.D.

5. Example. For the sequence $\xi_n = \{\sqrt{n}\}$ we have the bound $D_q = O(\sqrt{q})$.

6. **Osrtowski's theorem.** *Let $a = [a_0; a_1, ..., a_\nu, ...]$ be irrational number and there exists M such that all the partial quotients in its continued fraction are bounded by M:*

$$a_j \le M, \ \ \forall j$$

Then the discrepancy of the sequence $\{\alpha n\}, n = 1, 2, 3, ...$ satisfy

$$D_q \le 100M \log q, \ \ \forall q.$$

To prove this theorem we need Ostrowski's numerical system. Let q_ν be the sequence of the denominators of convergent fractions to $\alpha = [0; a_1, a_2, ...]$. Then every positive integer q can be written in a form

$$q = b_0 q_0 + b_1 q_1 + b_2 q_2 + ... + b_t q_t, \ \ \ b_j \le a_{j+1}.$$

7. **Weyl Criteria.**
a. **Theorem 1.** *The sequence ξ_j, $j = 1, 2, 3, ...$ is U.D. if and only if for any continuous function $f : [0, 1] \to \mathbb{R}(\mathbb{C})$ one has*

$$\lim_{q\to\infty} \frac{1}{q} \sum_{j=1}^{q} f(\xi_j) = \int_0^1 f(x)dx.$$

b. **Theorem 2.** *The sequence* ξ_j, $j = 1, 2, 3, ...$ *is U.D. if and only if for any* $m \in \mathbb{Z} \setminus \{0\}$ *one has*

$$\lim_{q\to\infty} \frac{1}{q} \sum_{j=1}^{q} e^{2\pi i m \xi_j} = 0, \quad e^{ix} = \cos x + i \sin x.$$

We will not prove the second criteria as we do not know Weierstrass theorem that any periodic continuous function on $[0, 1]$ can be approximated by a trigonomentic polynomial, that is

$$\forall\, f(x) \text{ continuous and periodic on } [0,1] \forall\, \varepsilon$$

$$\exists\, P(y) - \text{ polynomial, such that } \sup_{x\in[0,1]} |f(x) - P(e^{ix})| \le \varepsilon.$$

8. **Roth-Schmidt theorem.** *There exists an absolute constant c such that for any infinite sequence*

$$\limsup_{n\to\infty} \frac{D_q}{\log q} > c.$$

Exercises.

1. Obtain a good bound for the discrepancy of the sequence $\xi_n = \{\sqrt[3]{n}\}$.

2 a. Is $\xi_n = \{\log n\}$ U.D. or not?
b. For which β the sequence $\xi_n = \{(\log n)^\beta\}$ is U.D.?

3. Inverse to 4. Prove that if the sequence is U.D. Then $D_q = o(q)$, $q \to \infty$.

4. Van der Corput seuqence. For $n \ge 1$ consider the dyadic expansion

$$n - 1 = \sum_{j=0}^{s} a_j 2^j.$$

now we define

$$\xi_n = \sum_{j=0}^{s} \frac{a_j}{2^{j+1}}.$$

Prove that for this sequence $D_q = O(\log q)$.

5. Construct α such that the sequence $\{\alpha n!\}$ is U.D.