Bernoulli numbers

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Triangular numbers

Everyone is familiar with the sum

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

These numbers are called the triangular numbers.

For example,

$$\begin{aligned}
1 &= \frac{1(1+1)}{2} = 1 \\
1+2 &= \frac{2(2+1)}{2} = 3 \\
1+2+3 &= \frac{3(3+1)}{2} = 6 \\
1+2+3+4 &= \frac{4(4+1)}{2} = 10 \\
1+2+3+4+5 &= \frac{5(5+1)}{2} = 15
\end{aligned}$$

Pyramidal numbers

Many of you will also be familiar with the pyramidal numbers: these are the sums of squares:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

For example,

$$\begin{aligned}
1^2 &= \frac{1(1+1)(2+1)}{6} = 1 \\
1^2 + 2^2 &= \frac{2(2+1)(4+1)}{6} = 5 \\
1^2 + 2^2 + 3^2 &= \frac{3(3+1)(6+1)}{6} = 14 \\
1^2 + 2^2 + 3^2 + 4^2 &= \frac{4(4+1)(8+1)}{6} = 30 \\
1^2 + 2^2 + 3^2 + 4^2 + 5^2 &= \frac{5(5+1)(10+1)}{6} = 55
\end{aligned}$$

Generalizations?

Is there a general story here?

If k is a natural number larger than 2, is there a formula for the sum of the kth powers of the first n natural numbers?

$$1^k + \cdots + n^k = ???$$

Nicomachus

Nicomachus (Νικο′μαχος) was a Hellenistic mathematician who lived between around 60 CE and 120 CE in the Roman city of Gerasa, east of the Jordan River.

He was a *Neopythagorean*, who believed in religious doctrines based on the Greek philosophers Pythagorus and Plato. Among their beliefs, they associated God with the number One.

Nicomachus's Theorem

Nicomachus gave a formula for the sum of cubes:

$$1^3 + 2^3 + \cdots + n^3 = \big(1 + 2 + \cdots + n\big)^2$$

This is a beautiful formula but it doesn't really give an idea of how to understand the general sums

$$1^k + \cdots + n^k = ???$$

The proof uses mathematical induction.

For $n = 1$, both sides of the equation

$$1^3 + 2^3 + \cdots + n^3 = \big(1 + 2 + \cdots + n\big)^2$$

equal 1.

Suppose we have proved that

$$1^3 + 2^3 + \cdots + n^3 = \big(1 + 2 + \cdots + n\big)^2$$

We want to prove that replacing n by $n + 1$, both sides increase by the same amount, namely $(n + 1)^3$.

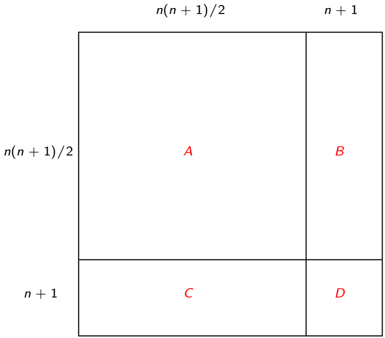
Consider the square with sides

$$1 + 2 + \cdots + n + (n + 1) = \frac{(n + 1)(n + 2)}{2},$$

and subdivide the sides into two segments, of length

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$$

and $n + 1$ respectively.



We see that

$$
\begin{aligned}
(1+2+\cdots+n+(n+1))^2 &= A+B+C+D \\
&= (1+2+\cdots+n)^2 + \frac{n(n+1)^2}{2} + \frac{n(n+1)^2}{2} + (n+1)^2 \\
&= (1^3+2^3+\cdots+n^3) + n(n+1)^2 + (n+1)^2 \\
&= (1^3+2^3+\cdots+n^3) + (n+1)^3.
\end{aligned}
$$

Johannes Faulhaber

The next steps in this subject were taken by Johannes Faulhaber (1585-1635), a mathematician who was born and lived in the German city of Ulm.

Faulhaber's Theorem

Faulhaber showed that for any fixed natural number k, the sum of powers

$$\Sigma_k(n) = 1^k + 2^k + \cdots + n^k,$$

thought of as a function of n, is a polynomial of degree $k+1$.

We have already seen this explicitly for $k = 0$, 1, 2 and 3, where the polynomials are respectively n, $n(n+1)/2$, $n(n+1)(2n+1)/6$, and $n^2(n+1)^2/4$.

In other words, there are numbers $b_{k,i}$ for $i = 1, \ldots, k+1$, such that

$$1^k + 2^k + \cdots + n^k = b_{k,k+1}n^{k+1} + b_{k,k}n^k + \cdots + b_{k,2}n^2 + b_{k,1}n.$$

It turns out that all of the coefficients of these polynomials are rational numbers.

(The constant term $b_{k,0}$ is absent because both sides vanish when $n = 0$.)

When we take $k = 4$, something interesting happens — the coefficient of the linear term n has quite a large denominator.

$$\Sigma_4(n) = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$$

Let's prove this special case of Faulhaber's Theorem: the general case is actually proved in exactly the same way.

Suppose that the left-hand side is a polynomial in n of degree 5:

$$\Sigma_4(n) = b_{4,5}n^5 + b_{4,4}n^4 + b_{4,3}n^3 + b_{4,2}n^2 + b_{4,1}n$$

If we can find values of $b_{4,k}$ such that both sides are equal for $n = 0$, and the right-hand side jumps by n^4 as we go from $n - 1$ to n, then the formula will be proved.

The case $n = 0$ is easy: both sides vanish.

It remains to check that

$$\left(b_{4,5}n^5 + b_{4,4}n^4 + b_{4,3}n^3 + b_{4,2}n^2 + b_{4,1}n\right) \\ - \left(b_{4,5}(n-1)^5 + b_{4,4}(n-1)^4 + b_{4,3}(n-1)^3 + b_{4,2}(n-1)^2 + b_{4,1}(n-1)\right) = n^4.$$

The left-hand side equals

$$b_{4,5}(n^5 - (n-1)^5) + b_{4,4}(n^4 - (n-1)^4) + b_{4,3}(n^3 - (n-1)^3) + b_{4,2}(n^2 - (n-1)^2) + b_{4,1}(n - (n-1)) \\ = b_{4,5}(5n^4 - 10n^3 + 10n^2 - 5n + 1) + b_{4,4}(4n^3 - 6n^2 + 4n - 1) + b_{4,3}(3n^2 - 3n + 1) + b_{4,2}(2n - 1) + b_{4,1}$$

The induction step consists of showing that the numbers $b_{4,i}$ may be chosen in such a way that this equals n^4.

To do this, it suffices to solve the simultaneous linear equations

$$\begin{aligned} 5b_{4,5} &= 1 \\ -10b_{4,5} + 4b_{4,4} &= 0 \\ 10b_{4,5} - 6b_{4,4} + 3b_{4,3} &= 0 \\ -5b_{4,5} + 4b_{4,4} - 3b_{4,3} + 2b_{4,2} &= 0 \\ b_{4,5} - b_{4,4} + b_{4,3} - b_{4,2} + b_{4,1} &= 0. \end{aligned}$$

It is easy to see that these have a unique solution, in which each coefficient $b_{4,i}$ is a rational number:

$$(b_{4,5}, b_{4,4}, b_{4,3}, b_{4,2}, b_{4,1}) = (1/5, 1/2, 1/3, 0, -1/30).$$

Two things about this solution are striking: the quadratic term is missing (that is, $b_{4,2} = 0$), and the linear term has the peculiar coefficient $-1/30$.

Let us list the coefficients $b_{k,i}$ in a table for $k = 1, \ldots, 6$:

k	$b_{k,1}$	$b_{k,2}$	$b_{k,3}$	$b_{k,4}$	$b_{k,5}$	$b_{k,6}$	$b_{k,7}$
0	1						
1	$1/2$	$1/2$					
2	$1/6$	$1/2$	$1/3$				
3	0	$1/4$	$1/2$	$1/4$			
4	$-1/30$	0	$1/3$	$1/2$	$1/5$		
5	0	$-1/12$	0	$5/12$	$1/2$	$1/6$	
6	$1/42$	0	$-1/6$	0	$1/2$	$1/2$	$1/7$

The numbers in the left column have a name: they are the **Bernoulli numbers** B_k.

Along the diagonals, this table shows some nice patterns. A little experimentation shows that the following conjectural formula holds explains all of the diagonal patterns in the table:

$$b_{k,i} = \frac{k!}{i!} \frac{B_{k-i+1}}{(k-i+1)!}.$$

Jakob Bernoulli

Jakob Bernoulli (1645–1705) was a member of a famous family of mathematicians in the Swiss city of Basel. He wrote an important book *Ars Conjectandi* (The Art of Conjecture) on probability and games of chance, and discovered the number e.

Jakob Bernoulli gave the first description of the numbers now named after him. In modern language, the numbers B_k are defined recursively by

$$k = \binom{k+1}{1} B_1 + \cdots + \binom{k+1}{k-2} B_{k-2} + \binom{k+1}{k-1} B_{k-1} + \binom{k+1}{k} B_k.$$

$1 = 2\,B_1$, so that $B_1 = 1/2$

$2 = 3\,B_1 + 3\,B_2$, so that $B_2 = 1/6$

$3 = 4\,B_1 + 6\,B_2 + 4\,B_3$, so that $B_3 = 0$

$4 = 5\,B_1 + 10\,B_2 + 10\,B_3 + 5\,B_4$, so that $B_4 = -1/30$

It is not hard to calculate a few more of these numbers for yourself. Eventually, you reach the bizarre number $B_{12} = -691/2730$: this hints that the Bernoulli numbers present a richer structure than most sequences of numbers you will have met in mathematics.

The Bernoulli numbers have many remarkable properties.

Theorem

If $k > 1$ is odd, then $B_k = 0$.

To see this, we extend the function $\Sigma_k(n)$ to negative values of n. Observe that $\Sigma_k(n)$ is completely determined by the formulas $\Sigma_k(0) = 0$ and $\Sigma_k(n) - \Sigma_k(n-1) = n^k$. This makes it natural to set

$$\Sigma_k(-n) = (-n+1)^k + (-n+2)^k + \cdots + (-1)^k.$$

This gives the remarkable identity

$$\Sigma_k(-n) = (-1)^k \Sigma_k(n-1).$$

If k is odd, we can calculate $\Sigma_k(n) + \Sigma_k(-n)$ in two different ways: if k is odd, the above identity shows that

$$\Sigma_k(n) + \Sigma_k(-n) = \Sigma_k(n) - \Sigma_k(n-1) = n^k.$$

On the other hand, Faulhaber's Theorem shows that

$$\Sigma_k(n) + \Sigma_k(-n) = 2\left(b_{k,k} n^k + b_{k,k-2} n^{k-2} + b_{k,k-4} n^{k-4} + \cdots + b_{k,1} n\right).$$

It follows that if $k > 1$, $B_k = b_{k,1} = 0$ (and $B_1 = 1/2$, but we already knew this).

Theorem

The signs of the even Bernoulli numbers oscillate: $B_2, B_6, \ldots,$ *are positive, and* $B_4, B_8, \ldots,$ *are negative.*

There are two ways to prove this result.

The first is as a corollary of the following theorem.

Theorem

The derivative

$$\left.\frac{d^{k-1}\tan(x)}{dx^{k-1}}\right|_{x=0}$$

equals $(-1)^{k/2-1}2^k(2^k-1)B_k/k$*. It vanishes if* k *is odd, and is a positive integer if* k *is even.*

The proof of this theorem uses differential calculus and mathematical induction: it depends on the ordinary differential equation

$$\frac{d\tan(x)}{dx} = 1 + \tan^2(x).$$

The second proof is as a consequence of a formula due to Euler. If $s > 1$, the infinite sum

$$\zeta(s) = 1^{-s} + 2^{-s} + 3^{-s} + \dots$$

converges. (This is shown using the comparison theorem from integral calculus.)

This sum is Riemann's zeta-function. It is an important tool in the study of prime numbers, because of Euler's product formula

$$\frac{1}{\zeta(s)} = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\dots\left(1 - \frac{1}{p^s}\right)\dots$$

The infinite product here is over all prime numbers. This formula is actually equivalent to the unique factorization theorem for the integers.

Obviously, $\zeta(s) > 1$; applying the comparison theorem from integral calculus to the definition of $\zeta(s)$, it also follows that $\zeta(s) < \frac{s}{s-1}$.

If $k > 0$ is even, Euler showed that

$$2\,\zeta(k) = -(2\pi i)^k \frac{B_k}{k!}.$$

This shows that the absolute value of the even Bernoulli numbers grows very fast as k increases: roughly as fast as k!

The Bernoulli number $B_{14} = 7/6$ is the first one such that $|B_k| > 1$. If you didn't have the idea of calculating out at least this far, you would never guess that the Bernoulli numbers increase rapidly in size with increasing k.

This shows the importance of old-fashioned calculation in mathematics: the great mathematicians are often the ones who have the patience and skill to calculate out further than the rest of us. A good example is Euler.

Proof of Bernoulli's formula

Theorem

For any fixed natural number k, the sum of powers

$$\Sigma_k(n) = 1^k + 2^k + \cdots + n^k,$$

thought of as a function of n, is the following polynomial in n of degree $k+1$:

$$\frac{1}{k+1}\left\{n^{k+1} + \binom{k+1}{1}B_1 n^k + \cdots + \binom{k+1}{k-1}B_{k-1}n^2 + \binom{k+1}{k}B_k n\right\}.$$

The proof uses the binomial theorem and mathematical induction.

We start with a trick:

$$\begin{aligned}
\Sigma_{k+1}(n) + (n+1)^{k+1} - 1^{k+1} &= 2^{k+1} + \cdots + (n+1)^{k+1} \\
&= (1+1)^{k+1} + \cdots + (n+1)^{k+1} \\
&= \left(1 + \tbinom{k+1}{1} + \cdots + \tbinom{k+1}{k} + 1\right) \\
&\quad + \left(1 + \tbinom{k+1}{1}2 + \cdots + \tbinom{k+1}{k}2^k + 2^{k+1}\right) \\
&\quad + \dots \\
&\quad + \left(1 + \tbinom{k+1}{1}n + \cdots + \tbinom{k+1}{k}n^k + n^{k+1}\right) \\
&= \Sigma_0(n) + \tbinom{k+1}{1}\Sigma_1(n) + \cdots + \tbinom{k+1}{k}\Sigma_k(n) + \Sigma_{k+1}(n).
\end{aligned}$$

Cancelling the common term $\Sigma_{k+1}(n)$ from both sides, we obtain the formula

$$(n+1)^{k+1} - 1 = \Sigma_0(n) + \tbinom{k+1}{1}\Sigma_1(n) + \cdots + \tbinom{k+1}{k}\Sigma_k(n).$$

By the induction hypothesis, we know that Bernoulli's formula holds for $j < k$:

$$\Sigma_j(n) = \tfrac{1}{j+1}\left(n^{j+1} + \tbinom{j+1}{1}B_1 n^j + \tbinom{j+1}{2}B_2 n^{j-1} + \cdots + \tbinom{j+1}{j}B_j n\right).$$

Inserting this into the previous formula, and expanding the left-hand side using the binomial theorem, we see that

$$\begin{aligned}
n^{k+1} + \tbinom{k+1}{k}n^k + \cdots + \tbinom{k+1}{1}n &= n \\
&+ \tfrac{1}{2}\tbinom{k+1}{1}\left(n^2 + \tbinom{2}{1}B_1 n\right) \\
&+ \ldots \\
&+ \tfrac{1}{k}\tbinom{k+1}{k-1}\left(n^k + \tbinom{k}{1}B_1 n^{k-1} + \tbinom{k}{2}B_2 n^{k-2} + \cdots + \tbinom{k}{k-1}B_{k-1} n\right) \\
&+ \left(n^{k+1} + \tbinom{k+1}{1}b_{k+1,k}n^k + \tbinom{k+1}{2}b_{k+1,k-1}n^{k-1} + \cdots + \tbinom{k+1}{k}b_{k,1}n\right)
\end{aligned}$$

Let us illustrate how this equation establishes the induction step, by comparing the coefficients of n on both sides:

$$\binom{k+1}{1} = 1 + \tfrac{1}{2}\binom{k+1}{1}\binom{2}{1}B_1 + \cdots + \tfrac{1}{k}\binom{k+1}{k-1}\binom{k}{k-1}B_{k-1} + \binom{k+1}{k}b_{k,1}.$$

A little massaging (rewrite the binomial coefficients in terms of factorials and make several cancellations) identifies this with the equation defining B_k in terms of the lower Bernoulli numbers:

$$k = \binom{k+1}{1}B_1 + \cdots + \binom{k+1}{k-2}B_{k-2} + \binom{k+1}{k-1}B_{k-1} + \binom{k+1}{k}b_{k,1}.$$

The proof of Bernoulli's formula for $b_{k,i}$ when $i > 1$ is similar, but it's best to try to work out the details yourself.

The Von Staudt-Clausen Theorem

The following remarkable theorem was proved independently by the German mathematicians Von Staudt and Clausen in 1840. (This theorem was not in my talk at the Tsinghua Math Camp, but I thought some of you would enjoy seeing it.)

Theorem

For every $k > 0$, the number

$$B_k + \text{sum of all fractions } \frac{1}{p} \text{ such that } p \text{ is a prime and } (p-1)|k$$

is an integer.

Remember that the symbol $(p-1)|k$ means that k is divisible by $p-1$.

For example,

$$\begin{array}{lll} B_1 + \frac{1}{2} = 1 & B_2 + \frac{1}{2} + \frac{1}{3} = 1 & B_4 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 \\ & B_6 + \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = 1 & B_8 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} = 1 \\ & B_{10} + \frac{1}{2} + \frac{1}{3} + \frac{1}{11} = 1 & B_{12} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} = 1 \\ & B_{14} + \frac{1}{2} + \frac{1}{3} = 2 & B_{16} + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = -6 \end{array}$$

We had to go quite far before the right-hand side equals anything other than 1.

Stirling numbers of the second kind

The first step in the proof seems to be a bit of a detour. Consider the set of all equivalence relations on the finite set $\{1, \ldots, k\}$. That is, we divide this set up into non-empty subsets, called parts, whose union is the whole set, and which have empty intersection with each other.

If $k = 3$, we have equivalence relations

$$
\begin{gathered}
123 \\
12|3, 13|2, 23|1 \\
1|2|3.
\end{gathered}
$$

If $k = 4$, we have the equivalence relations

$$
\begin{gathered}
1234 \\
123|4, 124|3, 134|2, 234|1, 12|34, 13|24, 23|14 \\
12|3|4, 13|2|4, 14|2|3, 23|1|4, 24|1|3, 34|1|2 \\
1|2|3.
\end{gathered}
$$

Each equivalence relation on $\{1, \ldots, k\}$ has a finite number of parts j, which lies between 1 and k.

Definition

The Stirling number of the second kind $S(k, j)$ is the number of equivalence relations on $\{1, \ldots, k\}$ with j parts.

We see that $S(k, j)$ is a strictly positive if $1 \leq j \leq k$, and vanishes otherwise.

The product $x^{\underline{j}} = x(x-1)\dots(x-j+1)$ is known as the falling power of x. When x is a natural number, it is related to the binomial coefficient by the formula

$$x^{\underline{j}} = j!\binom{x}{j}.$$

Theorem

If n and k are natural numbers, then

$$n^k = S(k,k)n^{\underline{k}} + S(k,k-1)n^{\underline{k-1}} + \cdots + S(k,1)n^{\underline{1}}.$$

Note that both sides of this formula are polynomials of degree k. Any two polynomials of degree k which agree at $k+1$ different values are equal. (Their difference is a polynomial of degree at most k, so has at most k roots unless it is equal to the zero polynomial.)

This shows that both sides are actually equal as polynomials in x:

$$x^k = S(k,k)x^{\underline{k}} + S(k,k-1)x^{\underline{k-1}} + \cdots + S(k,1)x^{\underline{1}}.$$

To prove this formula, we count the number of functions from the set $\{1, \ldots, k\}$ to the set $\{1, \ldots, n\}$ in two different ways. First of all, there are obviously n^k such functions.

Each such function may also be specified by the following four independent pieces of data:

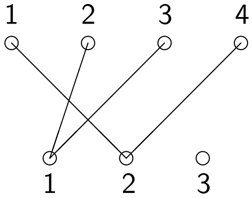
1. a natural number j between 1 and k — this is the number of elements of $\{1, \ldots, n\}$ in the image of the function;
2. an equivalence relation on $\{1, \ldots, k\}$ with j parts — this determines which elements of $\{1, \ldots, k\}$ map to the same element of $\{1, \ldots, n\}$;
3. a total order of the j parts of the equivalence relation;
4. a subset of $\{1, \ldots, n\}$ of cardinality j.

We may count the number of different assignments of these dats: it is the sum over j from 1 to k of terms of the form

$$S(k,j)j!\binom{n}{j} = S(k,j)n^{\underline{j}}.$$

Equating these two ways of counting the same set of functions, we obtain the desired identity.

For example, the function from $\{1,2,3,4\}$ to $\{1,2,3\}$ given by the picture



Here, $j = 2$, and the partition is $14|23$.

The sum of powers

$$1^k + \cdots + n^k$$

is quite complicated.

On the other hand, it is quite easy to calculate the sum of falling powers:

$$1^{\underline{k}} + \cdots + n^{\underline{k}} = \frac{(n+1)^{\underline{k+1}}}{k+1}.$$

Here is the whole proof:

$$\begin{aligned}
&(n+1)^{\underline{k+1}} - n^{\underline{k+1}} \\
&\quad = (n+1)n(n-1)\ldots(n-k+1) - n(n-1)\ldots(n-k+1)(n-k) \\
&\quad = ((n+1)-(n-k))\, n(n-1)\ldots(n-k+1) \\
&\quad = (k+1)n^{\underline{k}}.
\end{aligned}$$

Combining this formula with the formula for the power n^k as a combination of falling powers, we obtain the following identity.

Theorem

$$\Sigma_k(n) = \frac{S(k,k)}{k+1}(n+1)^{\underline{k+1}} + \frac{S(k,k-1)}{k}(n+1)^{\underline{k}} + \cdots + \frac{S(k,1)}{2}(n+1)^{\underline{2}}$$

Replacing n by $n-1$, and adding n^k to both sides, we obtain a related identity

$$\begin{aligned}\Sigma_k(n) &= \Sigma_k(n-1) + n^k \\ &= S(k,k)\frac{n^{\underline{k+1}}}{k+1} + S(k,k-1)\frac{n^{\underline{k}}}{k} + \cdots + S(k,1)\frac{n^{\underline{2}}}{2} + n^k.\end{aligned}$$

The linear term in n in the polynomial $n^{\underline{j}}$ is $(-1)^{j-1}(j-1)!$. If $k>1$, we may extract the linear terms in n on both sides of this identity: on the left-hand side, we obtain the Bernoulli number B_k.

Theorem

$$B_k = \sum_{j=1}^{k} (-1)^j \frac{j!}{j+1} S(k,j)$$

We will use this expression to prove the Von Staudt-Clausen Theorem.

We will also need the following formula for the Stirling numbers of the second kind:

$$j!S(k,j) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k.$$

To prove this, we insert the expression for i^k involving Stirling numbers and falling powers into the right-hand side.

$$\begin{aligned}
\sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} i^k &= \sum_{i=0}^{j} \sum_{\ell=0}^{k} (-1)^{j-i} \binom{j}{i} S(k,\ell) i^{\underline{\ell}} \\
&= \sum_{\ell=0}^{k} \ell! S(k,\ell) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} \binom{i}{\ell} \\
&= \sum_{\ell=0}^{k} \ell! S(k,\ell) \binom{j}{\ell} \sum_{i=0}^{j} (-1)^{j-i} \binom{j-\ell}{j-i}.
\end{aligned}$$

To finish the proof, we use that, by the binomial theorem,

$$\begin{aligned}\sum_{i=0}^{j}(-1)^{j-i}\binom{j-\ell}{j-i} &= \sum_{i=0}^{j}(-1)^{i}\binom{j-\ell}{i} \\ &= (1-1)^{j-\ell} \\ &= \begin{cases}1, & j=\ell, \\ 0, & j>\ell.\end{cases}\end{aligned}$$

This formula is very useful for calculating $S(k,j)$ for small values of j:

$$S(k,2) = 2^{k-1} - 1$$

$$S(k,3) = \frac{1}{2}\left(3^{k-1} + 1\right) - 2^{k-1}.$$

In particular, we see that if k is even, $S(k,3)$ is even:

$$3^{k-1} + 1 \equiv 0 \pmod 4.$$

We can now prove the Von Staudt-Clausen theorem.

As a first step, let us show that if k is even and $j+1$ is composite, then

$$\frac{j!}{j+1}S(k,j)$$

is an integer. There are three cases:

1. If $j+1 = ab$ is composite and $a > b$, then $\frac{(ab-1)!}{ab}$ is an integer.
2. If $j+1 = a^2$ is composite and $a > 2$, then $\frac{(a^2-1)!}{a^2}$ is an integer. Since $S(k,j)$ is always an integer, this completes the proof of the assertion in these two cases.
3. If $j+1 = 4$, then $S(k,3)$ is even, and hence

   $$\frac{j!}{j+1}S(k,j) = \frac{3}{2}S(k,3)$$

   is an integer.

In this way, we see that only primes $j+1=p$ contribute to the expression for the Bernoulli number, modulo integers:

$$B_k + \sum_{p \text{ prime}} \frac{1}{p} \sum_{i=0}^{p-1} (-1)^{p-i} \binom{p-1}{i} i^k \in \mathbb{Z}.$$

Note that $i^k \equiv i^{k-(p-1)} \pmod{p}$. This means that in calculating

$$\sum_{i=0}^{p-1} (-1)^{p-i} \binom{p-1}{i} i^k \pmod{p},$$

we may as well assume that k lies between 0 and p, by subtracting multiples of $p-1$ from k.

If $(p-1) \nmid k$, then we obtain

$$\sum_{i=0}^{p-1} (-1)^{p-i} \binom{p-1}{i} i^k \equiv -(p-1)! S(k, p-1) \equiv 0 \pmod{p},$$

since we may take $k < p-1$, and in this case the Stirling number vanishes.

If $(p-1) \mid k$, then we obtain

$$\sum_{i=0}^{p-1} (-1)^{p-i} \binom{p-1}{i} i^{p-1} \equiv \sum_{i=1}^{p-1} (-1)^{p-i} \binom{p-1}{i} \equiv (-1)^{p-1} \pmod{p},$$

completing the proof.

Hirzebruch's Formula

Friedrich Hirzebruch (1927–2012) was of one of Germany's great postwar mathematicians.

I want to close this talk with my favorite formula involving Bernoulli numbers, which was found by Hirzebruch in his thesis. This formula is of great importance in algebraic geometry, where it lies behind formulas for counting the number of solutions of algebraic equations.

Theorem (Hirzebruch)

The coefficient of x^n in the polynomial

$$\left(1 + B_1 x + B_2 \frac{x^2}{2!} + \cdots + B_n \frac{x^n}{n!}\right)^{n+1}$$

is 1.

The proof uses the residue theorem of complex analysis. In fact, this formula gives another characterization of the Bernoulli polynomials.

Thankyou for being such a good audience!