Probability and Statistics
Tsinghua Math Camp Prof. Paul Horn
Homework 2: Due Friday 7/21/2016

Note: You should get to know your coaches, Gavin St. John (`gavin.stjohn@du.edu`) and Zhe Liu (刘喆) (`z-liu16@mails.tsinghua.edu.cn`). If you need to contact me, my email address is `paul.horn@du.edu` or you may ask via WeChat.

I want very much to learn all of your names, but please be patient with me as I am not so good at pronouncing Chinese and I am very bad at remembering names (even non-Chinese names). So please help me by telling me your name very slowly and being patient with me as I try to learn and pronounce it.

Warm-up problems: These are simple problems to make sure you follow basic definitions. You need not turn them in.

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Homework problems: To be turned in.

1. Suppose, for the purpose of this question, that children of a given pair of parents are uniformly likely to be boys or girls, uniformly likely to born on any day of the week, and that the gender of different children of the same parents are equally likely to be boys and girls.

   A couple have two children.

   (a) What is the probability that the younger child is a boy, if the older child is a boy?

   (b) What is the probability that both children are boys, given that one of the children is a boy?

   (c) What is the probability that both children are boys given that one is a boy born on Tuesday?

   (d) Explain the difference between the three situations.

2. Give an example showing that mutual independence and pairwise independence are different concepts.

3. 100 people board an airplane with 100 seats; the ith person to board has a ticket for seat number i. The first person to board, however, is a jerk and sits in a uniformly random seat (potentially his own – but only with probability $\frac{1}{100}$.) Each other person sits in their own seat if it's free, but otherwise sits in a random seat. Find the probability that the last person to board sits in their own seat.
   Too hard? Start by trying to solve if if there are only 2 or 3 or 4 people on the airplane.
   Too easy? Generalize to n seats. Find the probability that the kth from the last person sits in their own seat for $1 \leq k < n$.

4. Suppose $A_1, A_2, \dots$ are a sequence of events. Let $\mathcal{A}_\infty$ denote the event that infinitely many of the A_i occur. Let $p_i = \mathbb{P}(A_i)$.

   (a) Prove that if $\sum p_i < \infty$,
   $$\mathbb{P}\big(\mathcal{A}_\infty\big) = 0.$$
   Note/Hint: One way of proving that $\mathbb{P}(\cdot) = 0$ is to prove that $\mathbb{P}(\cdot) < \epsilon$ for any positive number ϵ.

   (b) **Challenge:** Prove that if the events $(A_i)_{i=1}^\infty$ are mutually independent and $\sum p_i = \infty$, then
   $$\mathbb{P}\big(\mathcal{A}_\infty\big) = 1.$$

   (c) **Even if you can't do (b):** Give an example showing that the hypothesis of part (b) cannot be removed in general. That is, find events $(A_i)_{i=1}^\infty$ in some probability space so that $\sum p_i = \infty$, but so that $\mathbb{P}(\mathcal{A}_\infty) < 1$.