**Probability and Statistics**

Tsinghua Math Camp Prof. Paul Horn

**Homework 1:** Due Wednesday 7/19/2017

**Note:** You should get to know your coaches, Gavin St. John (`gavin.stjohn@du.edu`) and Zhe Liu (刘喆) (`z-liu16@mails.tsinghua.edu.cn`). If you need to contact me, my email address is `paul.horn@du.edu` or you may ask via WeChat.

I want very much to learn all of your names, but please be patient with me as I am not so good at pronouncing Chinese and I am very bad at remembering names (even non-Chinese names). So please help me by telling me your name very slowly and being patient with me as I try to learn and pronounce it.

**Homework problems:** To be turned in.

1. An integer is chosen uniformly at random from $\{1, \dots, 10\}$.
   (a) Describe the set of outcomes $\Omega$
   (b) Describe each of the following events as a set:
   - The number is even.
   - The number is prime.
   - The number is even and prime.
   - The number is even or prime.
   - The number is not even and is not prime.

   (c) Find the probability of each of these events.

2. You are given a 6 sided dice.
   (a) Describe a method to use the dice (possibly rolled multiple times) generate a uniformly random integer in $\{1, 2, \dots, 12\}$.
   **Note:** Rolling the dice twice and adding the result gives a number between 1 and 12, but it is *not* uniform.
   (b) Describe a method to use the dice (possibly rolled multiple times) to generate a uniformly random number in $\{1, \dots, 5\}$.
   (c) **Challenge:** Describe a method to use the dice to generate a uniformly random number in $\{1, \dots, n\}$ for an arbtirary $n$.

3. A match of tennis is made up of a number of sets (where each set is comprised of a number of games.) In the men's singles draw at Wimbledon, the final match is a best-of-five format: that is, the first player who wins 3 sets (out of a maximum of 5) wins.
   (a) Suppose Roger Federer has a 60% chance of winning any given set over Martin Cilić. Find the probability that he wins the final. (What assumtions are you making?)

(b) Find an expression for the probability of Federer winning a best of $n$ sets match for general $n$. Is it better for Federer (the stronger player) for the match to consist of more sets or fewer.

4. Suppose $A_1, A_2, A_3, \ldots$ are events in a probability space.

   (a) Prove

$$\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup \ldots) \geq \sum_{i=1}^{\infty} \mathbb{P}(A_i) - \left( \sum_{1 \leq i < j < \infty} \mathbb{P}(A_i \cap A_j) \right).$$

   (b) Suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ and let $\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$. Prove

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(\lim_{n\to\infty} A_n)$$

5. Which would you prefer and why: I spin a wheel with the numbers $0-100$ and give you that many thousands RMB, or I spin a wheel with the numbers $0-10$ ten times, add up the results and give you that many thousands RMB. Justify your answer. (There is no 'right answer' here, but defend why you think your method is better.) Note that the 'expected' winnings in both scenarios is the same (50,000 RMB.)

**Additional Warm-up problems:** Want some extra practice? These are simple problems to make sure you follow basic definitions. You need not turn them in. You might even start with them.

1. Five fair coins are flipped. Describe the set of outcomes $\Omega$, and describe the event that there are an odd number of heads as a subset of $\Omega$.

2. A standard deck of cards has thirteen 'ranks' $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K)$ and four suits $\diamondsuit, \heartsuit, \clubsuit, \spadesuit$. Each card has one suit and one rank, and there is one of each so there are $4 \times 13 = 52$ cards in a deck.

   Five cards are chosen from a shuffled (random) deck. The order the cards are drawn does not matter.

   (a) What is the probability that a three of a kind is drawn. That is, three of one rank, and two of different ranks. (So, for instance $K\heartsuit, K\spadesuit, K\clubsuit, 3\heartsuit, 2\diamondsuit$ is an example but $K\heartsuit, K\spadesuit, K\clubsuit, 3\heartsuit, 3\diamondsuit$ is not, nor is $K\heartsuit, K\spadesuit, K\clubsuit, K\diamondsuit, 2\diamondsuit$ )

   (b) What is the probability that two pairs is drawn. That is, two cards of one rank, two cards of a different rank, and one card of a third rank. (eg. $K\heartsuit, K\diamondsuit, 6\heartsuit, 6\clubsuit, 2\diamondsuit$).

3. Does it make sense for take a uniformly random element of a countably infinite set? Why or why not?

4. Does it make sense to take a random element of a countably infinite set? Describe a probability measure on such a space?