LINEAR ALGEBRA HOMEWORK 2 - DUE 7/26

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Assume U, V, W are F-vector spaces.

Exercise 0.1. *As before, let* $\mathrm{Hom}(U, V)$ *be the set of* F*-linear maps* $U \to V$*. How would you make it an* F*-vector space? Prove your assertions.*

Exercise 0.2. *If* $f : U \to V$ *and* $g : V \to W$ *are linear maps, show that their composition* $gf \equiv g \circ f : U \to W$ *is also linear.*

Exercise 0.3. *WRITE UP Let* $\mathrm{Iso}(V, V)$ *be the set of isomorphisms (i.e. linear bijections)* $\phi : V \to V$*, and* $\mathrm{Iso}(V, W)$ *the set of isomorphisms* $f : V \to W$*. Suppose* $f_0 \in \mathrm{Iso}(V, W)$*. Show that the map*

$$T : \mathrm{Iso}(V, V) \to \mathrm{Iso}(V, W), \ \phi \mapsto f_0 \circ \phi$$

bijects.

Exercise 0.4. *WRITE UP Obviously, we can transform* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *to* $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *by a single R1 operation. Show that you can achieve the same with a sequence of operations involving only R2 and R3 types. More generally, show that on any matrix, an R1 operation can always be replaced by a sequence of operations of types R2 and R3. Try to do it in 10 lines (or less).*

Exercise 0.5. *Describe the complete set of solutions* S_0 *to the following system in* $\mathbb{R}^4$*, by giving an isomorphism* $f : \mathbb{R}^k \to S_0$ *(including specifying the appropriate* k*):*

$$\begin{aligned} x + y + z + t &= 0 \\ x + y + 2z + 2t &= 0 \\ x + y + 2z - t &= 0. \end{aligned}$$

Replace the 0 *on the right side of first equation by* 1*, and then describe the solution set* S_1 *of the new system. Find a 'nice' bijection (again as nice as you can think of)* $\mathbb{R}^k \to S_1$*. (Hint: Use a nice bijection* $S_0 \to S_1$*.*

Exercise 0.6. *WRITE UP Let $\mathcal{L}$ be an n-variable linear system over a field F. Prove that*

$$\begin{aligned} & \mathcal{L} \textit{ is homogenous} \\ \Leftrightarrow\ & 0 \in sol(\mathcal{L}) \\ \Leftrightarrow\ & sol(\mathcal{L}) \textit{ is } F\textit{-subspace of } F^n. \end{aligned}$$

Try to make your proof as simple as possible, say less than 1/2 a page.

Exercise 0.7. *Let $F[x]$ be the space of F-valued polynomial functions in one variable x. You saw that $F[x]$ is an F-space. Verify that the familiar operation of differentiation:*

$$\frac{d}{dx} : F[x] \to F[x]$$

from 'calculus', is F-linear.

Exercise 0.8. *Let $F[x]_d$ be the F-subspace of $F[x]$ consisting of all polynomials $p(x)$ of degree at most d, i.e. the highest power x^n appearing in $p(x)$ is at most x^d. Consider the linear map*

$$\frac{d}{dx} : F[x]_d \to F[x]_d.$$

Describe the F-subspaces $\ker \frac{d}{dx}$ and $\operatorname{im} \frac{d}{dx}$ of $F[x]_d$.
Hint: Experiment!