Research projects for Analysis/Topology
Tsinghua Mathcamp 2015

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1 Winding numbers and degrees

Recall that for any continuous path $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{P\}$, we can define its *winding number* $W(\gamma, P)$, which is roughly speaking the net change in angle around P along the path.

Problem 1.1. *Let γ and δ be two continuous paths from an interval to $\mathbb{R}^2 \setminus \{P\}$, which are either paths with the same endpoints or closed paths. Show that they are homotopic (through paths with the same endpoints or closed paths respectively) if and only if $W(\gamma, P) = W(\delta, P)$.*

Problem 1.2. *If $F : C \to C'$ and $G : C' \to C''$ are continuous maps, where C, C' and C'' are all circles, then what can you say about the relation among the degrees of F, G and $G \circ F$? Can you prove your answer?*

Problem 1.3. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear isomorphism, represented by a (2×2)-matrix. Show that the local degree of F at the origin is $+1$ is the determinant of the (2×2)-matrix is positive, and -1 if the determinant is negative.*

2 Applications of winding numbers

2.1 Fixed point property

Problem 2.1. *Which of the following spaces have the fixed point property?*

(i) *a closed rectangle;*

(ii) *the plane;*

(iii) *an open interval;*

(iv) *an open disk;*

(v) *a circle;*

(vi) *a sphere S^2;*

(vii) a torus $S^1 \times S^1$;

(viii) a solid torus $S^1 \times D^2$.

Justify your answers.

Problem 2.2. *Try to prove the higher dimensional version of the Brouwer Fixed Point Theorem: any continuous map from a closed ball $B \subset \mathbb{R}^n$ to itself must have a fixed point.*

Problem 2.3. *If a topological space X is compact and contractible, does X have the fixed point property? Give a proof if you think this is true, or find a counterexample if you think this is false.*

2.2 Antipodes

Problem 2.4. *Let $f : S \to S'$ be a continuous map between spheres. Show that if $f(P) \neq f(P^*)$ for all $P \in S$, then f must be surjective; here P^* denotes the antipode of P.*

Problem 2.5. *Prove that there is no continuous retraction from D^n onto S^{n-1} for an positive integer n.*

Problem 2.6. *Prove that there is no continuous map f from the n-dimensional sphere S^n to the sphere S^{n-1} such that $f(P^*) = f(P)^*$ for all P.*

3 De Rham cohomology

Problem 3.1. *Let $U = \mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}$, i.e. $\mathbb{R}^2$ minus n distinct points. Compute the 1st cohomology group $H^1(U)$.*

Problem 3.2. *For the sphere S^2 and torus $S^1 \times S^1$, compute all their cohomology groups. Can you do the same also for S^n and T^n for $n > 2$?*